Math 246C Lecture 2 Notes

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1 Holomorphic Curves in $\mathbb{C}^2$ and Holomorphic Functions on Riemann Surfaces

1.1 Holomorphic curves in $\mathbb{C}^2$

Last time, we were discussing complex tori.

Example 1.1 (complex tori)**.** We have $X = \mathbb{C}/\Lambda$, where Λ is a lattice. We have a natural quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$. Let V_1, V_2 be the images of two charts $\varphi_i : U_i \to V_i$, $i = 1, 2$. Consider $\varphi_2 \circ \varphi_1^{-1}(z) =: \psi(z)$. Then for $z \in \varphi_1(U_1 \cap U_2)$, $\pi|_{V_2}(\psi(z)) = \pi|_{V_1}(z)$, so $\psi(z) - z \in \Lambda$. Since Λ is discrete, $\psi(z) - z$ is locally constant. So it is holomorphic.

Here is another natural example of a Riemann surface.

Example 1.2 (holomorphic curves in $\mathbb{C}^2 = \mathbb{C}^2_{z,w}$)**.** Let $\Omega \subseteq \mathbb{C}^2$ be open, and let $f \in \mathrm{Hol}(\Omega)$; that is, $f \in C^1(\Omega)$, and $f(z, w)$ is separately holomorphic: $z \mapsto f(z, w)$ is holomorphic for all w and $w \mapsto f(z, w)$ is holomorphic for all z. We have the Cauchy-Riemann equations

$$\frac{\partial f}{\partial \overline{z}}(z, w) = 0, \qquad \frac{\partial f}{\partial \overline{w}}(z, w) = 0.$$

Assume that $(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}) \neq 0$ for all $(z, w) \in f^{-1}(\{0\})$.

We claim that $X = f^{-1}(\{0\})$ is a (possibly disconnected) Riemann surface. Let $(z_0, w_0) \in X$. If $f'_w(z_0, w_0) \neq 0$, then by the holomorphic implicit function theorem (which we will prove), there exist an open neighborhood $V \subseteq \mathbb{C}^2$ of (z_0, w_0), $z_0 \in U \subseteq \mathbb{C}$, and $g \in \mathrm{Hol}(U)$ such that $X \cap V = \{(z, g(z)) : z \in U\}$. So the projection $\pi_z : X \cap V \to U$ sending $(z, w) \mapsto z$ is a chart. Similarly, if $f'_z(z_0, w_0) \neq 0$, we have locally near (z_0, w_0): $X \cap V = \{(h(w), w)\}$, where h is holomorphic. So the projection $\pi_w : X \cap V \to \mathbb{C}$ is a chart. Compatibility of charts is the following diagram:

$$\begin{tikzcd} X \arrow[r, "\pi_w"] \arrow[d, "\pi_z"'] & U_w \\ U_z \arrow[ur, "\pi_w \circ \pi_z^{-1}"'] & \end{tikzcd}$$

Theorem 1.1 (holomorphic implicit function theorem)**.** *Let $f(z,w) : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic near $(0,0) \in \mathbb{C}^2$ with $f'(a,b) \neq 0$. Then $f = 0$ determines a holomorphic map $\varphi : \mathbb{C} \to \mathbb{C}$ in a neighborhood of (a,b).*

Proof. Let $f(z,w)$ be holomorphic near $(0,0) \in \mathbb{C}^2$ with $f(0,0) = 0$ and $f'_w(0,0) \neq 0$. Choose $r > 0$ so that $w \mapsto f(0,w)$ is holomorphic when $|w| < 2r$ and $f(0,w) \neq 0$ when $0 < |w| < 2r$. Then choose $\delta > 0$ such that f is holomorphic when $|w| < 3r/2$, $|z| < \delta$ and such that $f(z,w) \neq 0$ when $|w| = r$, $|z| < \delta$. By the argument principle, for $|z| < \delta$,

$$|\{w \in D(0,r) : f(z,w) = 0\}| = \frac{1}{2\pi i} \int_{|w|=r} \frac{f'_w(z,w)}{f(z,w)}\, dw,$$

where the right hand side is holomorphic in z. So for all z with $|z| < \delta$, the equation $f(z,w) = 0$ has exactly 1 root $w = w(z)$ in $D(0,r)$. Write

$$w(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{w f'_w(z,w)}{f(z,w)}\, dw, \qquad |z| < \delta$$

by the residue theorem. □

1.2 Holomorphic functions on Riemann surfaces

Definition 1.1. Let X be a Riemann surface equipped with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$. We say that $f : X \to \mathbb{C}$ is **holomorphic** if for all α, $f \circ \varphi_\alpha^{-1} \in \mathrm{Hol}(V_\alpha)$. Let Y be a Riemann surface equipped with an atlas $\{\varphi'_\beta : U'_\beta \to V'_\beta\}$. A continuous map $f : X \to Y$ is called **holomorphic** if for all α, β, $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(f^{-1}(U'_\beta) \cap U_\alpha) \to V'_\beta$ is holomorphic.

Theorem 1.2. *Let X, Y be Riemann surfaces, and let $f_j \in \mathrm{Hol}(X,Y)$, $j = 1,2$. Assume that there exists $A \subseteq X$ with a limit point $a \in X$ such that $f_1 = f_2$ on A. Then $f_1 \equiv f_2$.*

Proof. (Sketch) Use the connectedness of the Riemann surfaces to transplant the corresponding result from complex analysis. □

Proposition 1.1 (local normal form for $f \in \mathrm{Hol}(X,Y)$)**.** *Let X, Y be Riemann surfaces, and let $f_j \in \mathrm{Hol}(X,Y)$ be non-constant. Let $a \in X$. Then there exist complex charts $\varphi : U \to V$ on X with $a \in U$, $\varphi(a) = 0$ and $\psi : U' \to V$; on Y with $f(a) \in U'$, $\psi(f(a)) = 0$, $U \subseteq f^{-1}(U')$ such that the holomorphic function*

$$F = \psi \circ f \circ \varphi^{-1} : V \to V'$$

is of the form $F(z) = z^k$ for some $k \in \mathbb{N}^+$.

Remark 1.1. The integer k is independent is independent of the charts.

Proof. Take any charts φ, ψ centered at $a, f(a)$. Then $\tilde{F}(z) = (\psi \circ f \circ \varphi^{-1})(z) \in \mathrm{Hol}(\mathrm{neigh}(0, \mathbb{C}))$, and $\tilde{F}(0) = 0$. So $\tilde{F}(z) = z^k g(z)$, where g is holomorphic and non-vanishing. In a simply connected neighborhood of 0, there exists a holomorphic function $h \neq 0$ such that $g = h^k$. The map $\kappa(z) = zh(z)$ is a holomorphic diffeomorphism from $\mathrm{neigh}(0, \mathbb{C}) \to \mathrm{neigh}(0, \mathbb{C})$ by the inverse function theorem. Replace φ by $\kappa \circ \varphi$, we get $[\psi \circ f \circ (\kappa \circ \varphi)^{-1}](z) = z^k$. $\square$

We will discuss the integer k more next time.